QCD phase diagram with both fluctuation and finite coupling effects in the strong coupling lattice QCD

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A. Ohnishi, T. Ichihara and T. Z. Nakano, PoS LATTICE2012, 088 (2012),
T. I ., T. Z. Nakano, A. Ohnishi PoS Lattice2013, 143 (2013),
T. I., A. Ohnishi, T. Z. Nakano, arxiv:1401.4647

Finite density QCD and sign problem

- Finite density QCD
  - Neutron stars, Early universe,
    Heavy ion collisions (RHIC, LHC) ,…
  - QCD phase diagram, Critical point, Inhomogeneous structure, …
- Sign problem
  - In QCD, fermion det. becomes complex due to the chemical potential.
    -> breakdown of the probability interpretation
  - Approaches to finite μ region
    Reweighting Fodor, Katz (2002,04)…, Taylor expansion, Ejiri(2004),Bazavov,Ding,Hegde, Kaczmarek, Karsch, Laermann, Mukherjee, Petreczky, Schmidt, Smith *et al*. (2012)…, Imaginary μ de Forcrand, O. Philipsen (2002), D'Elia, Lombardo(2003)…, Canonical approach Ejiri (2008), Li, Alexandru, Liu, (2011)…, Complex Langevin Aarts, Seiler, Stamatescu (2010), Seiler, Sexty, Stamatescu (2013)…, Dual variables Mercado, Gattringer, Schmidt (2013)…, Lefschetz thimble Cristoforetti, Renzo, Scorzato [AuroraScience Collaboration] (2012), Fujii, Honda, Kato, Kikukawa, Komatsu, Sano (2013), Strong coupling…

Sign problem and representations

- **The sign problem and partition function**

$$Z = \mathrm{tr} e^{-\beta H} = \sum_n \langle n | e^{-\beta H} | n \rangle$$

  - | n > are eigen states of Hamiltonian : no sign problem
  - | n > are not eigen states of Hamiltonian : sign problem
  - **The sign problem depends on representation of the states.**

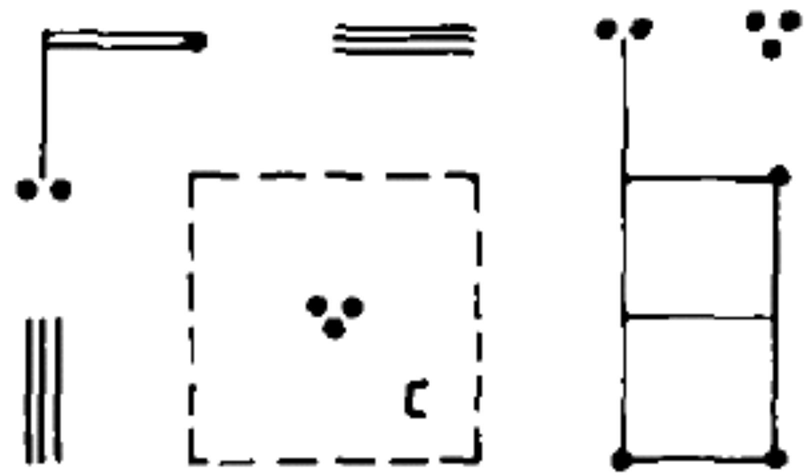
- **The representation**
  - How to alter representations?
    - **idea : converting integration procedure -> dual variables**

Dual variables - 1

- Bosonic systems — Endres, Gattringer, Schmidt, Azcoiti, …
- Fermionic systems — Karsch, Mutter (1988), Chandrasekharan (2006), de Forcrand, Fromm (2010), Unger, de Forcrand (2011)…
  - **Strong coupling lattice QCD (SC-LQCD)**
    - **Long history of study** (Wilson (1974), Creutz (1980), Munster (1981), Kawamoto, Smit, Faldt, Petersson,, Damggard,…)
    - **Strong coupling expansion (1/g2 expansion)**
      - Expansion in plaquette terms
    - **Integration procedure (different from standard Lattice QCD)**
      1. **link variables**
      2. **Grassmann variables**
    - **Weaker sign problem in SC-LQCD** compared with standard Lattice QCD
      - Effective action in terms of **hadronic d.o.f.**
        → We expect weaker sign problem in SC-LQCD.
      - No sign problem in the mean field (MF) approximation
      - Sign problem with fluctuation effects

Dual variables - 2

- Bosonic systems — Endres, Gattringer, Schmidt, Azcoiti, …
- Fermionic systems — Karsch, Mutter (1988), Chandrasekharan (2006), de Forcrand, Fromm (2010), Unger, de Forcrand (2011)
  - **Strong coupling lattice QCD (SC-LQCD)**
    Monomer-Dimer-Polymer (MDP) simulation
    - **Integration procedure**
      1. Link variables
      2. Grassmann variables
      3. Monomer-Dimer-Polymer configurations
    - **Representation**
      hadronic d.o.f.
    - **Characteristics**
      1. Exact transformation from lattice QCD action in the strong coupling limit
      2. Mechanism for weakening the sign problem
         - Resummention technique
      3. Auto correlation time
    - Worm algorithm
      - Strong coupling limit, Next-to-leading order by reweighing



Karsch, Mutter (1988)

de Forcrand, Fromm (2010), Unger, de Forcrand (2011), de. Forcrand et. al. (2013), Unger (2014)

Auxiliary field Monte-Carlo (AFMC) method

- Auxiliary field Method on QCD phase diagram in SC-LQCD
  - **Another way to convert representations in SC-LQCD**
  - **Integration procedure**
    1. Link variables
    2. Bosonization
    3. Grassmann variables
    (4. Auxiliary field configurations in AFMC)
  - **Representation**
    hadronic d.o.f.
  - **Characteristics**
    1. Manifest physical mode
    2. Manifest chiral symmetry
    3. Straightforward to include finite coupling effect
  - Mean field analysis
    - Strong coupling limit, next-to-leading order, and next-to-next-to-leading order effects
      Nishida (2004), Fukushima (2004), Miura, Nakano, Ohnishi, Kawamoto (2009), Nakano, Miura, Ohnishi (2011)
  - AFMC
    - Strong coupling limit

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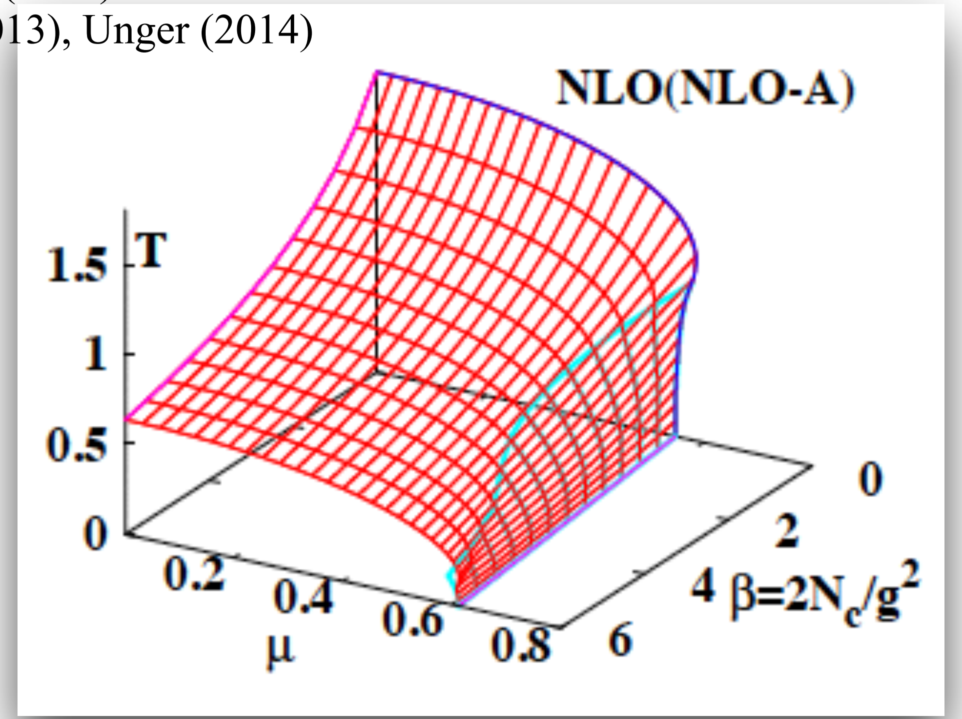
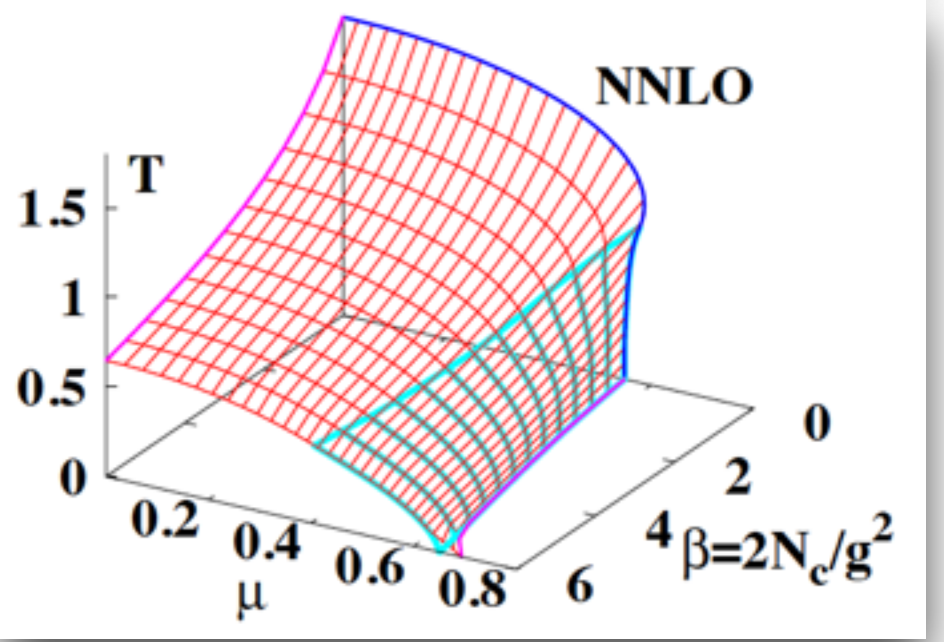
Finite coupling effects on QCD phase diagram in MF

MF : Miura, Nakano, Ohnishi, Kawamoto (2009)
Nakano, Miura, Ohnishi (2011)
Reweighting: de. Forcrand et. al. (2013), Unger (2014)

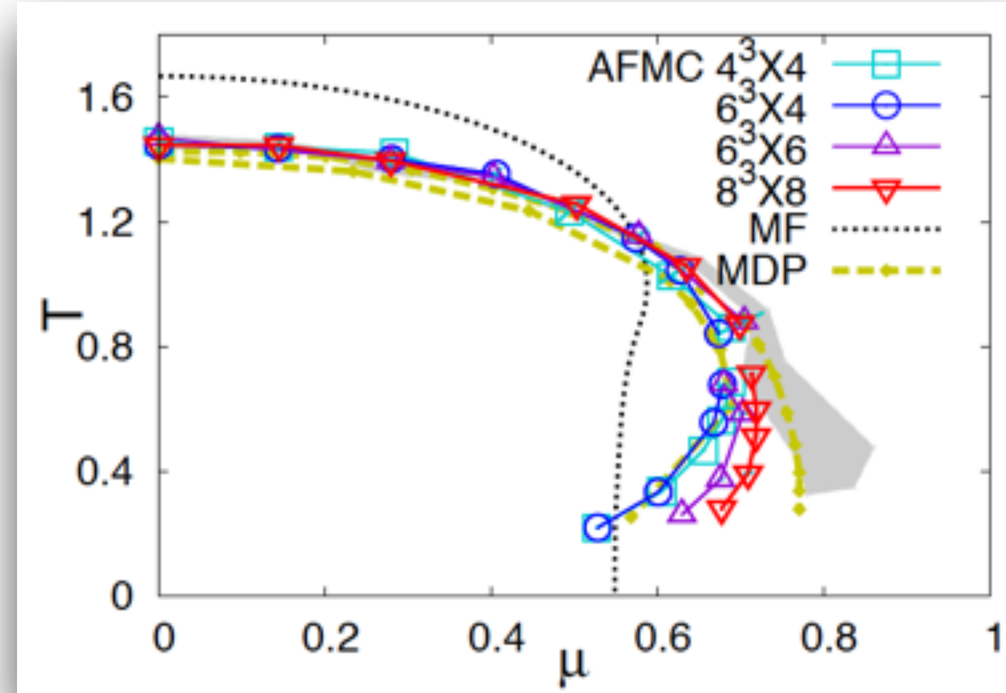
- Finite coupling effects
  - To obtain the insight into the continuum limit
    - QCD phase diagram evolution (1st. order phase line)
  - To evaluate the influence on Critical point
    - Density fluctuation can be included via NLO bosonization, which is important effects on QCD critical point. Fujii, Ohtani (2003.2004)



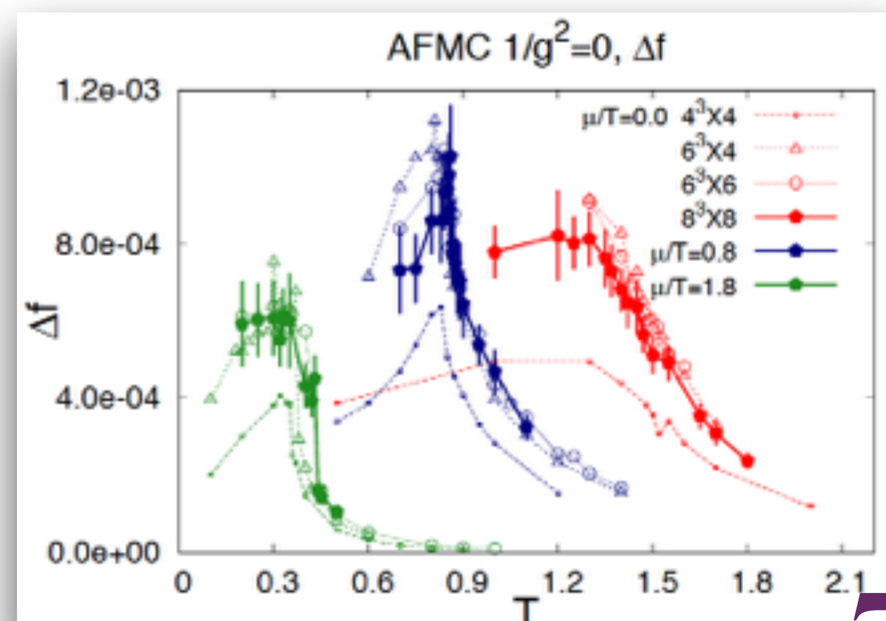


Strong coupling lattice QCD with fluctuations in the strong coupling limit (SCL)

- Fluctuation effects
  - Important step to evaluate partition function exactly
  - **Current numerical approaches**
    - Monomer-Dimer-Polymer (MDP) simulation
      Karsch, Mutter (1989,1990) . de Forcrand, Fromm (2010), Unger, de. Forcrand (2011)
    - Auxiliary field Monte-Carlo (AFMC) method
      T. I ., T. Z. Nakano, A. Ohnishi PoS Lattice2013, 143 (2013),
      T. I. , A. Ohnishi, T. Z. Nakano, arXiv:1401.4647 [hep-lat]
  - QCD phase diagram in SCL
  - Origin of sign problem
    - MDP :Baryon loop configurations
    - AFMC : Bosonization procedure



Δf : the difference of the free energy density between full and phase quenched simulation



Purpose

- To develop a method to include **both**

    1. **finite coupling**
    (Next-to-Leading order (NLO) of strong coupling expansion here)

    2. **fluctuation effects**

- To discuss the sign problem in AFMC

- To investigate phase diagram evolution

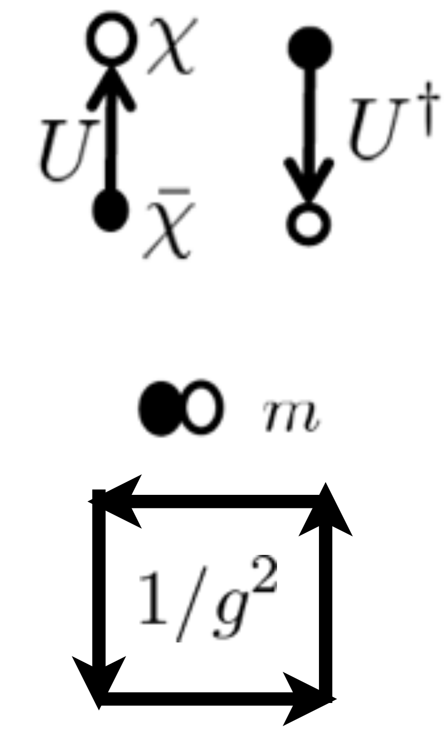
Lattice QCD action

- Unrooted staggered fermion, anisotropic lattice, lattice spacing a=1

$$S_{\mathrm{LQCD}} = \frac{1}{2}\sum_{x,\nu=0}^{d}\left[\eta^{+}_{\nu,x}\bar{\chi}_x U_{\nu,x}\chi_{x+\hat{\nu}} - \eta^{-}_{\nu,x}(\mathrm{H.C.})\right]$$

$$+\frac{m_0}{\gamma}\sum_x \bar{\chi}_x\chi_x$$

$$+\frac{2N_c\xi}{g_\tau^2(g_0,\xi)}\mathcal{P}_\tau + \frac{2N_c}{g_s^2(g_0,\xi)\xi}\mathcal{P}_s$$



- Assuming $\gamma = \xi$ and $g_\tau = g_s$, temporal lattice spacing is expressed as $a_\tau = a/\gamma^2$ due to quantum corrections, so we here define $T = \gamma^2/N_\tau a$.
  (T_c (μ=0) does not depend on aniso. parameters.)
  N. Bilic et. al. (1992, 1995)

$$\eta^{\pm}_{\nu,x} = (e^{\pm\mu a_\tau}, (-1)^{x_1\cdots x_{\nu-1}}/\gamma)$$

$$\mathcal{P}_i = \sum_{P_i}\left[1 - \frac{1}{2N_c}\mathrm{Tr}(U_{P_i} + U^\dagger_{P_i})\right]$$

U_{P_i} : plaquette term (i=τ, s)

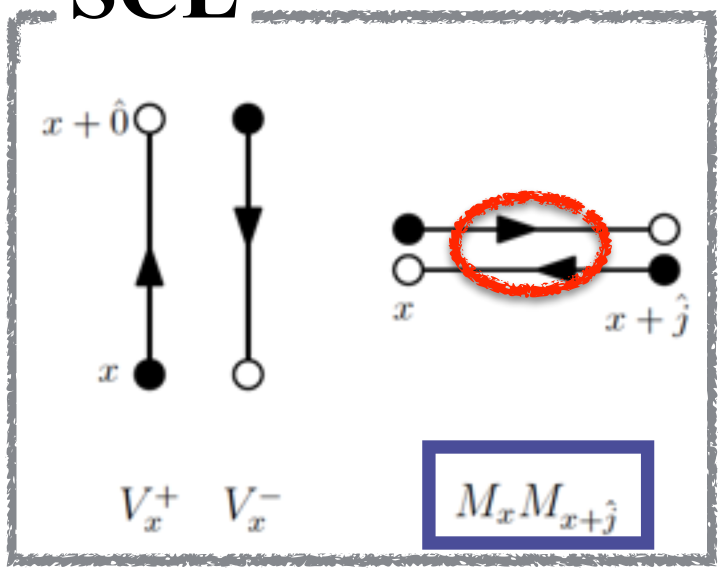
Effective action in the strong coupling limit

- $1/g^2$ expansion, leading order of $1/d$ (large dimensional) expansion

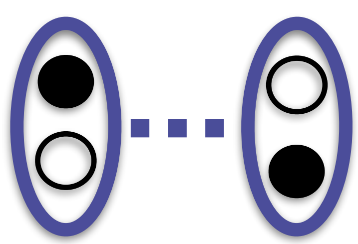
- ***U_j* (spatial link) integration**

$$\int dU\, U_{ab}\, U^\dagger_{cd} = \frac{1}{N_c}\delta_{ad}\,\delta_{bc}$$

SCL



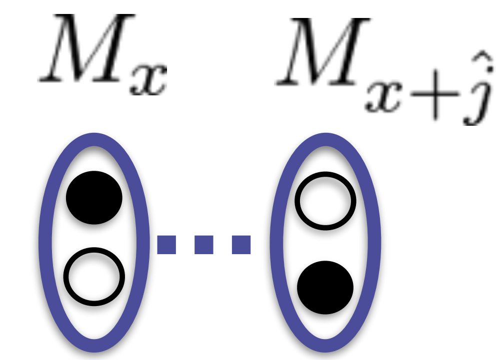
$$V_x^+ = e^{\mu a_\tau}\bar{\chi}_x U_{0,x}\chi_{x+\hat{0}}\ ,$$
$$V_x^- = e^{-\mu a_\tau}\bar{\chi}_{x+\hat{0}} U^\dagger_{0,x}\chi_x\ ,$$
$$M_x = \bar{\chi}_x \chi_x\ ,$$



$$S_{\text{eff}} = \frac{1}{2}\sum_x \left[V_x^+ - V_x^-\right] + \frac{1}{4N_c\gamma}\sum_{x,j} M_x M_{x+\hat{j}} + \frac{m_0}{\gamma}\sum_x M_x$$

Auxiliary filed Monte-Carlo (AFMC) method

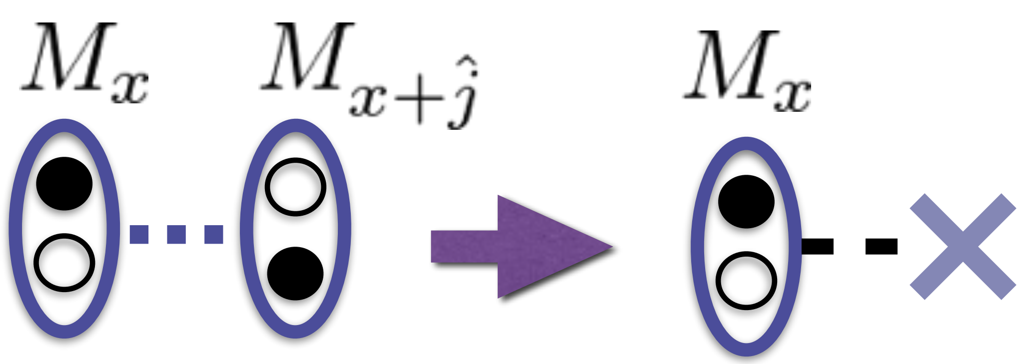
- **Extended HS (EHS) transformation**
  - Fluctuation effects : Different value at each site
  - Necessity to introduce complex term



$$\exp\left[\alpha AB\right] = \int \mathcal{D}\left[\phi,\varphi\right]\exp\left[-\alpha\left\{\phi^2+\varphi^2+(A+B)\varphi-\boxed{i}(A-B)\phi\right\}\right]$$

Complex coefficient

- Bosonization



AFs
σ_x
π_x

AFMC

$$Z = \int \mathcal{D}\boxed{\left[\sigma,\pi\right]}e^{-S_{\mathrm{eff}}(\sigma,\pi)}$$

Integration over AFs by Monte-Carlo technique

- Modified mass

$$m_x = m_0 + \frac{1}{4N_c}\sum_j(\sigma+i\epsilon\pi)_{x\pm\hat{j}} \qquad \epsilon_x = (-1)^{x_0+\cdots+x_d}$$

Effective action and AFMC method

- **Effective action (after Grassmann and U_0 integration) in SCL**

$$S_{\rm eff}^{\rm AF} = \sum_{\mathbf{k},\tau,f(\mathbf{k})>0} \frac{L^3 f(\mathbf{k})}{4N_c} \left[|\sigma_{\mathbf{k},\tau}|^2 + |\pi_{\mathbf{k},\tau}|^2\right] - \sum_{\mathbf{x}} \log\left[X_{N_\tau}(\mathbf{x})^3 - 2X_{N_\tau}(\mathbf{x}) + 2\cosh(3N_\tau\mu/\gamma^2)\right]$$

  - $X_{N_\tau} = X_{N_\tau}[m_x]$, $m_x = m + \frac{1}{4N_c}\sum_j (\sigma + i\epsilon\pi)_{x\pm\hat{j}}$
  - Smaller phase at larger μ

$$f(\boldsymbol{k}) = \sum_{j=1}^{d} \cos k_j$$

$$\epsilon_x = (-1)^{x_0+\cdots+x_d}$$

- Auxiliary filed Monte-Carlo (AFMC) method

AFMC

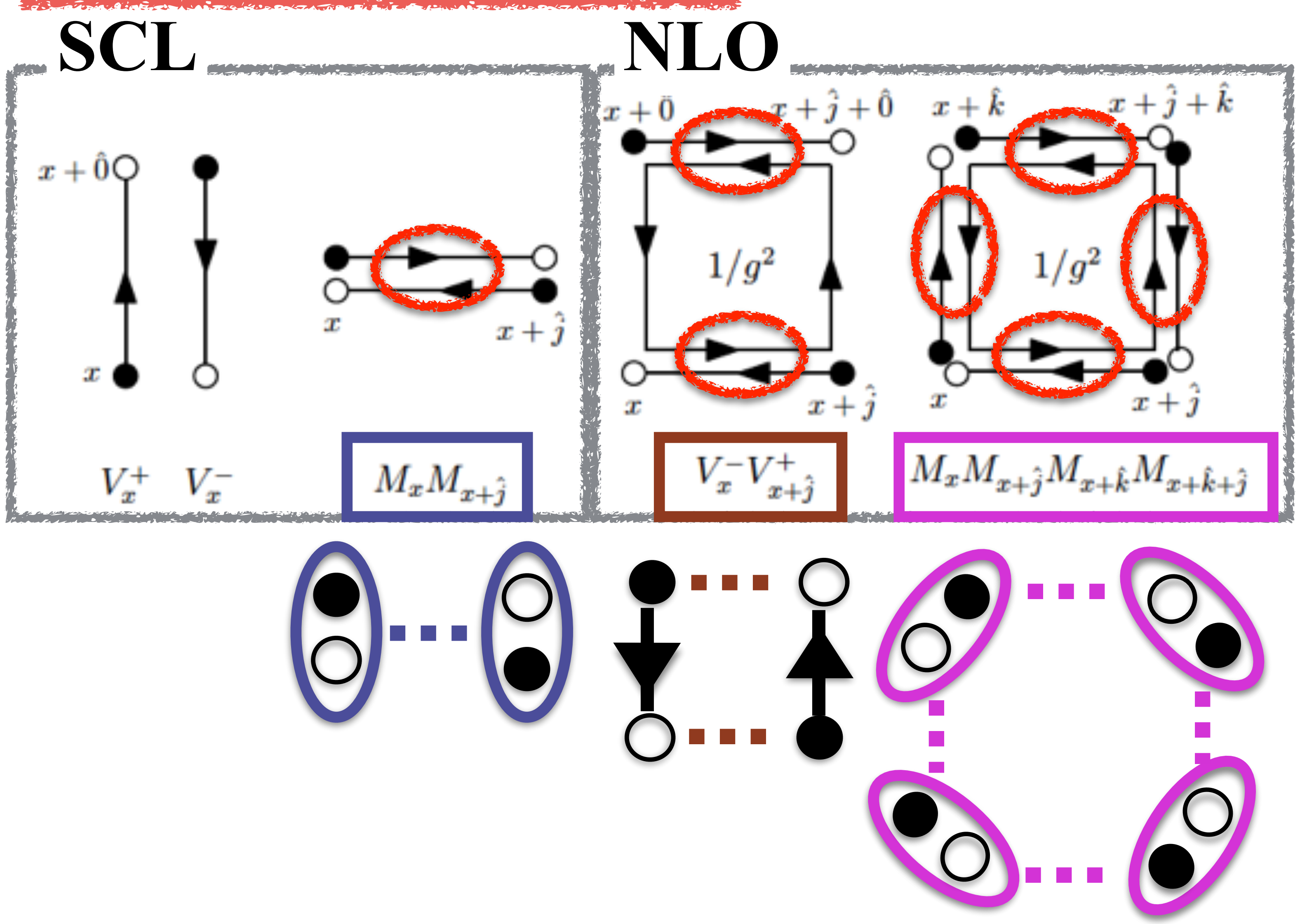
$$Z = \int \mathcal{D}\boxed{[\sigma,\pi]}\, e^{-S_{\rm eff}(\sigma,\pi)}$$

Integration over AFs by Monte-Carlo technique

Effective action with NLO terms

- $1/g^2$ expansion, leading order of $1/d$ (large dimensional) expansion
- U_j (spatial link) integration

$$\int dU\, U_{ab} U^\dagger_{cd} = \frac{1}{N_c}\delta_{ad}\,\delta_{bc}$$

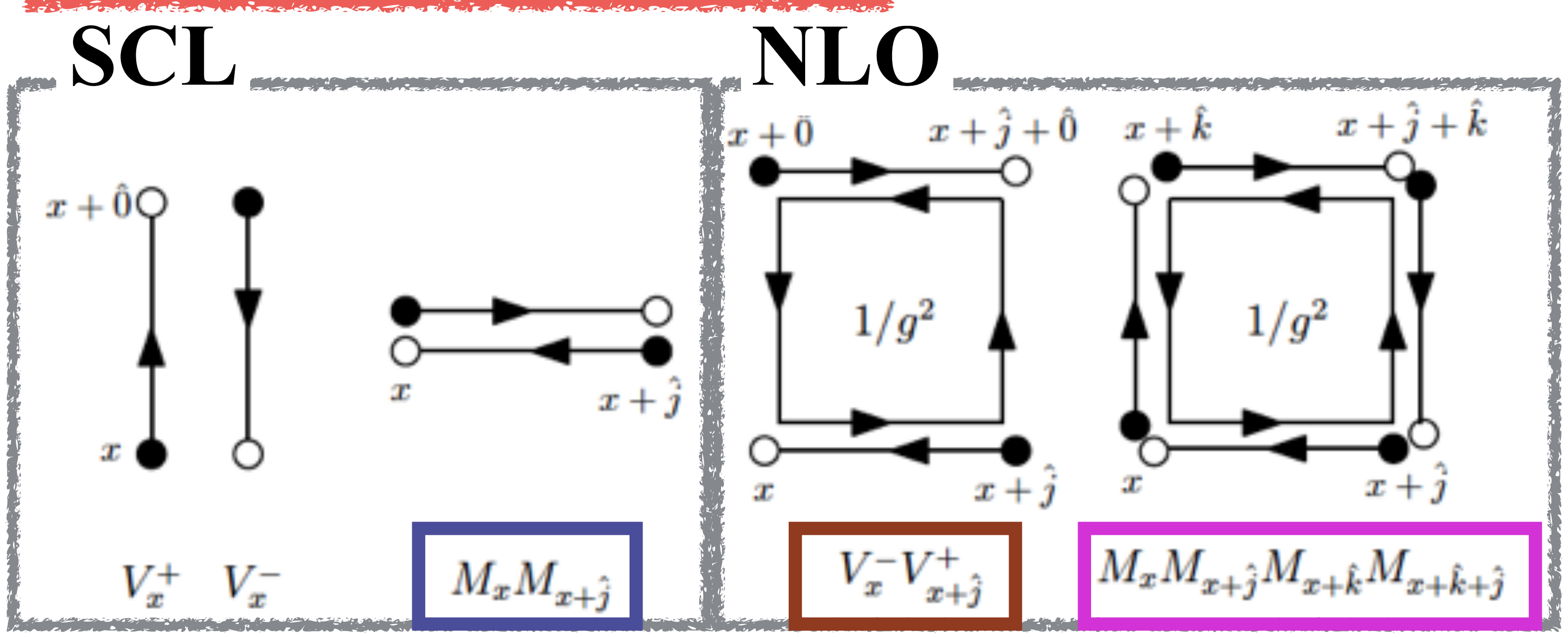


$$V^+_x = e^{\mu a_\tau}\bar{\chi}_x U_{0,x}\chi_{x+\hat{0}}\ ,$$
$$V^-_x = e^{-\mu a_\tau}\bar{\chi}_{x+\hat{0}} U^\dagger_{0,x}\chi_x\ ,$$
$$M_x = \bar{\chi}_x\chi_x\ ,$$

Effective action with NLO terms

- $1/g^2$ expansion, leading order of $1/d$ (large dimensional) expansion

- U_j (spatial link) integration



$$V_x^+ = e^{\mu a_\tau} \bar{\chi}_x U_{0,x} \chi_{x+\hat{0}} \ ,$$
$$V_x^- = e^{-\mu a_\tau} \bar{\chi}_{x+\hat{0}} U_{0,x}^\dagger \chi_x \ ,$$
$$M_x = \bar{\chi}_x \chi_x \ ,$$

- Extended Hubbard-Stratonovich (EHS) transformation
  - spatial terms ; *MMMM* $\rightarrow$ *MM* $\rightarrow$ M (sequential bosonization)
  - temporal terms ; *V V* $\rightarrow$ V

Origin of sign problem

- $$\exp\left[\alpha AB\right] = \int \mathcal{D}\left[\phi, \varphi\right] \exp\left[-\alpha\left[\phi^2 + \varphi^2 + (A+B)\varphi - i(A-B)\phi\right]\right]$$

Monte-Carlo integration (AFMC)

Effective action with NLO terms

- Effective action after bosonization(Φ are auxiliary fields (AFs), SCL=strong coupling limit, sp.=spatial, t.=temporal, NLO=next leading order)

$$S_{\text{eff}}^{\text{EHS}} = \frac{1}{2}\sum_x \Phi_x^2 + \sum_x m_x(\Phi) M_x + \frac{1}{2}\sum_x Z_x(\Phi)\left[V_x^+(\tilde{\mu}(\Phi)) - V_x^-(\tilde{\mu}(\Phi))\right]$$

$$V_x^+ = e^{\mu a_\tau}\bar{\chi}_x U_{0,x}\chi_{x+\hat{0}} \ ,$$
$$V_x^- = e^{-\mu a_\tau}\bar{\chi}_{x+\hat{0}} U_{0,x}^\dagger \chi_x \ ,$$
$$M_x = \bar{\chi}_x \chi_x \ ,$$

  - **modified mass** $m_0 \to m_x(\Phi_{\text{SCL}}, \Phi_{\text{sp.NLO}})$
  - **modified chemical potential** $\mu \to \tilde{\mu}_x(\Phi_{\text{t.NLO}})$
  - **wave function renormalization** $1 \to Z_x(\Phi_{\text{t.NLO}})$

- Grassmann & U_0 (temporal link) integration

- NLO effective action in terms of **hadronic d.o.f.**
  → Detail expressions are given in the back-up slides

- Auxiliary filed Monte-Carlo (AFMC) method
  We integrate out auxiliary fields by Monte-Carlo technique

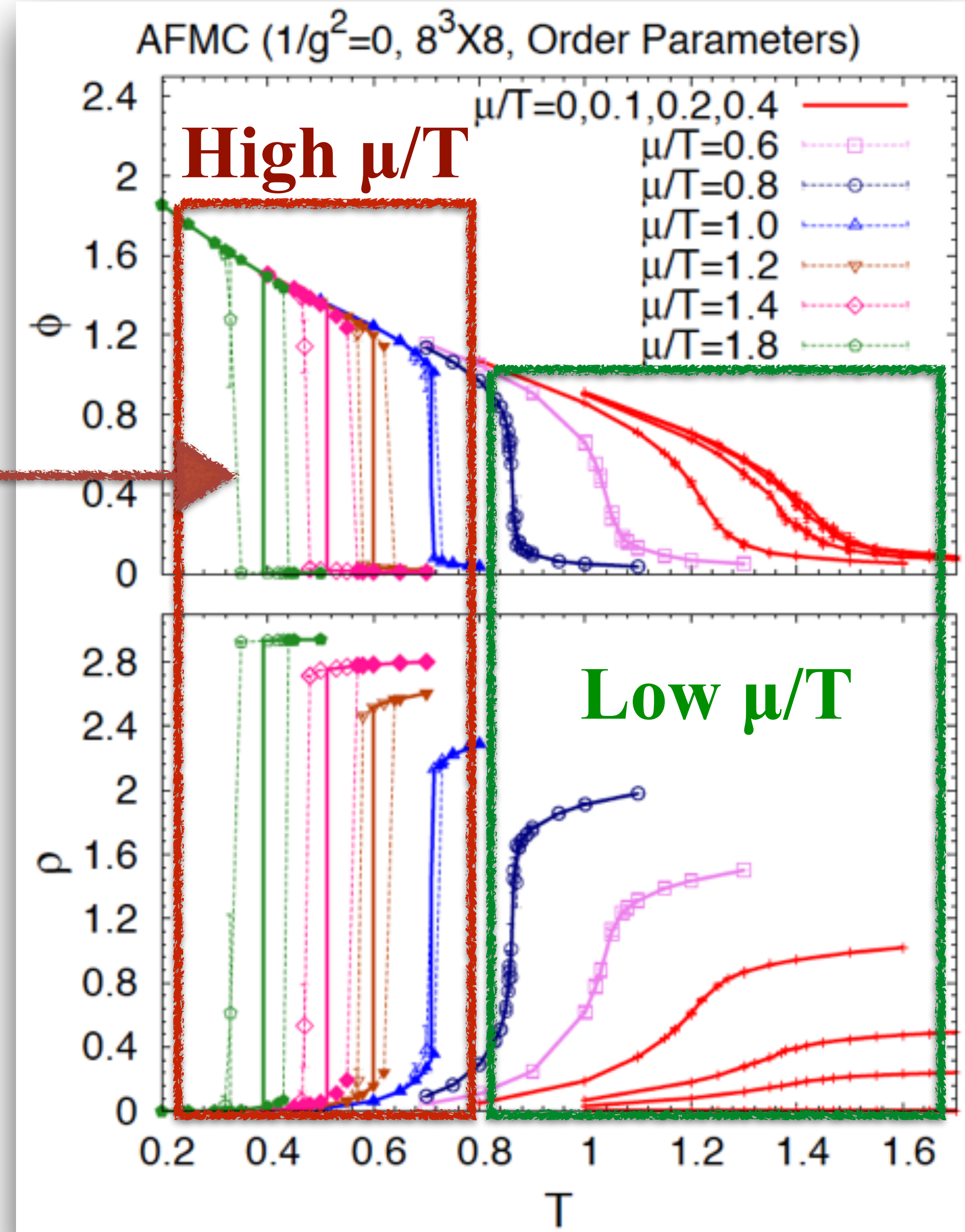
Results and discussion

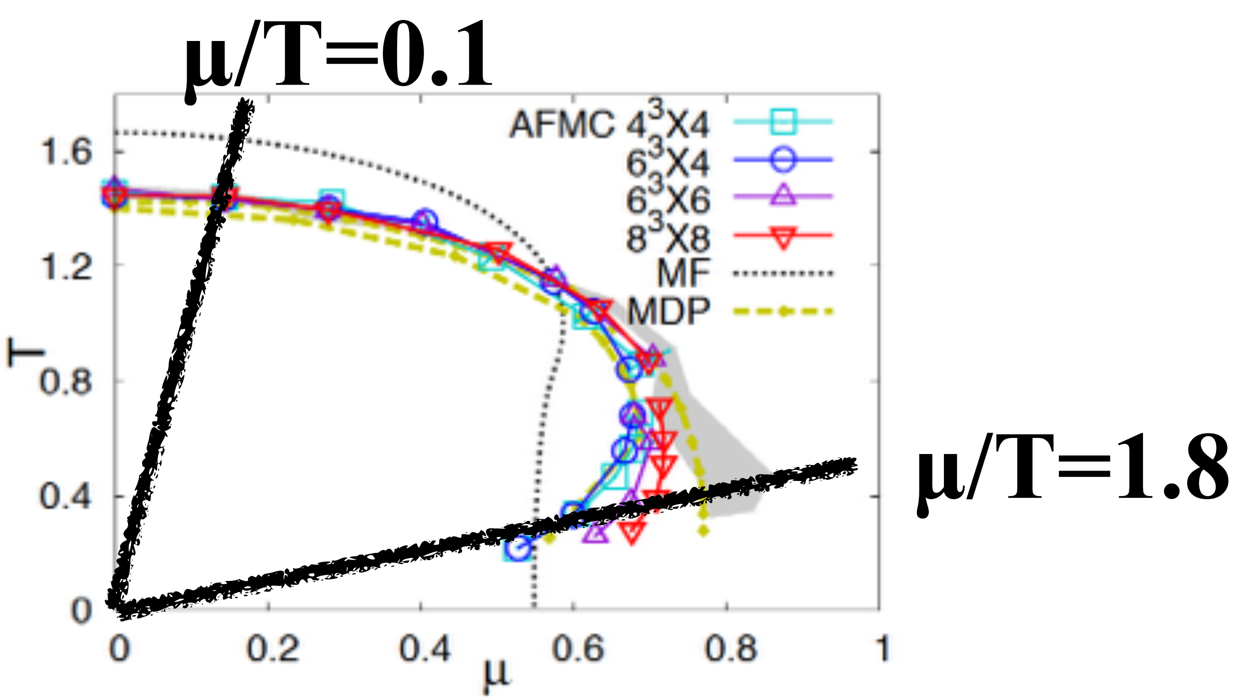
- Reservation
  - Unrooted staggered fermion (n_f=4 in the continuum limit)
  - Anisotropic lattice
  - chiral limit
  - all results are shown in the lattice unit
  - We show results of
    SCL
    t.NLO (SCL + temp. plaq. NLO terms)
    sp.NLO (SCL + sp. palq. NLO terms)

strong coupling limit (SCL)
next-to-leading order (NLO)

Results - strong coupling limit (SCL)

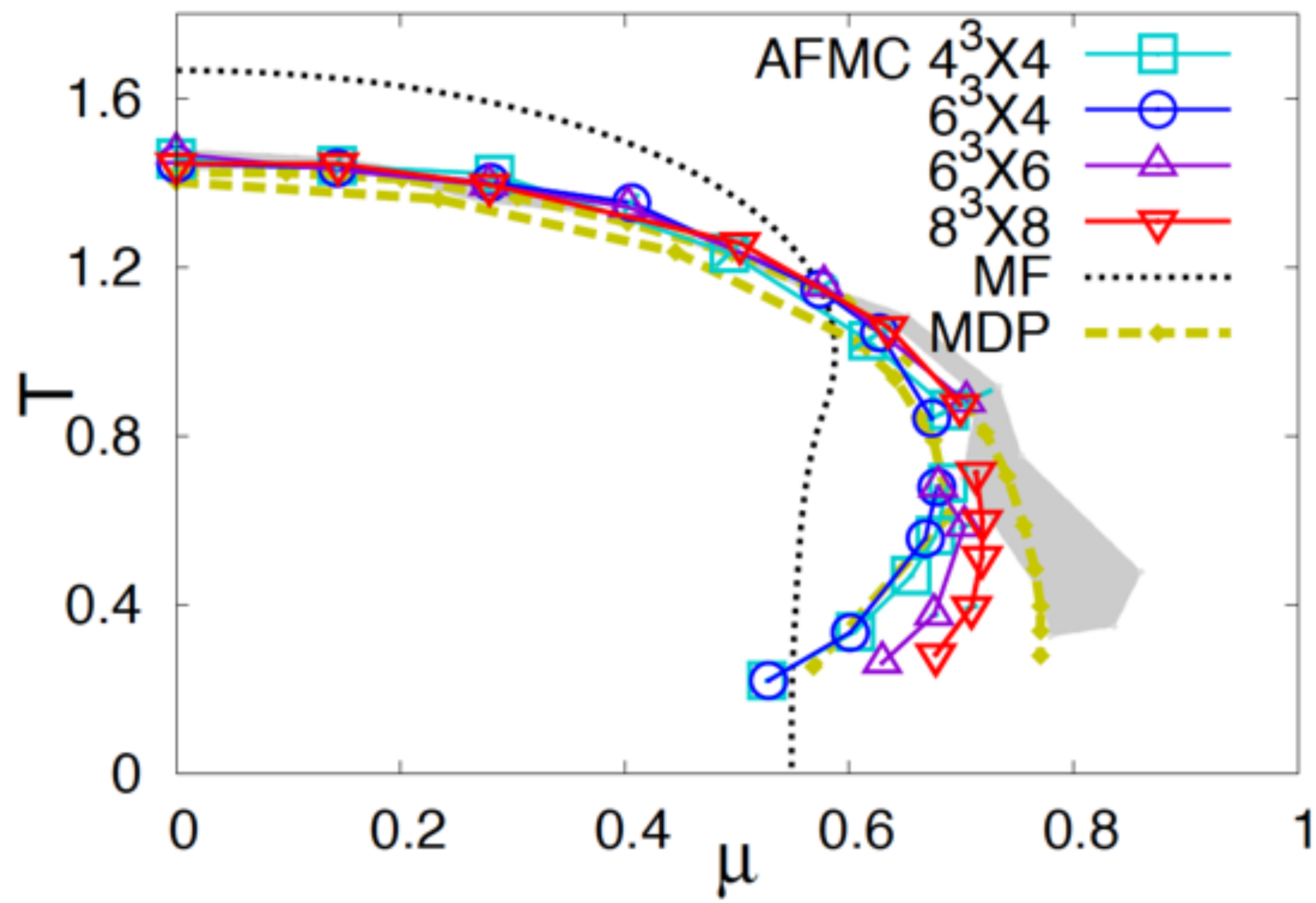
- Low μ/T
    - 2nd order or crossover (would-be second)
- High μ/T
    - 1st order (would-be first)
        - hysteresis
        - dependence on initial conditions
          Wigner start (σ = 0.01) and NG start (σ=2)





Results - phase diagram in SCL

- Low μ/T
  - Chiral susceptibility peak
  - Reduced Tc
  - almost no size dependence
- High μ/T
  - Comparing with effective action from different initial conditions
  - Enhanced μc
  - small spatial size dependence
  - Nτ dependence



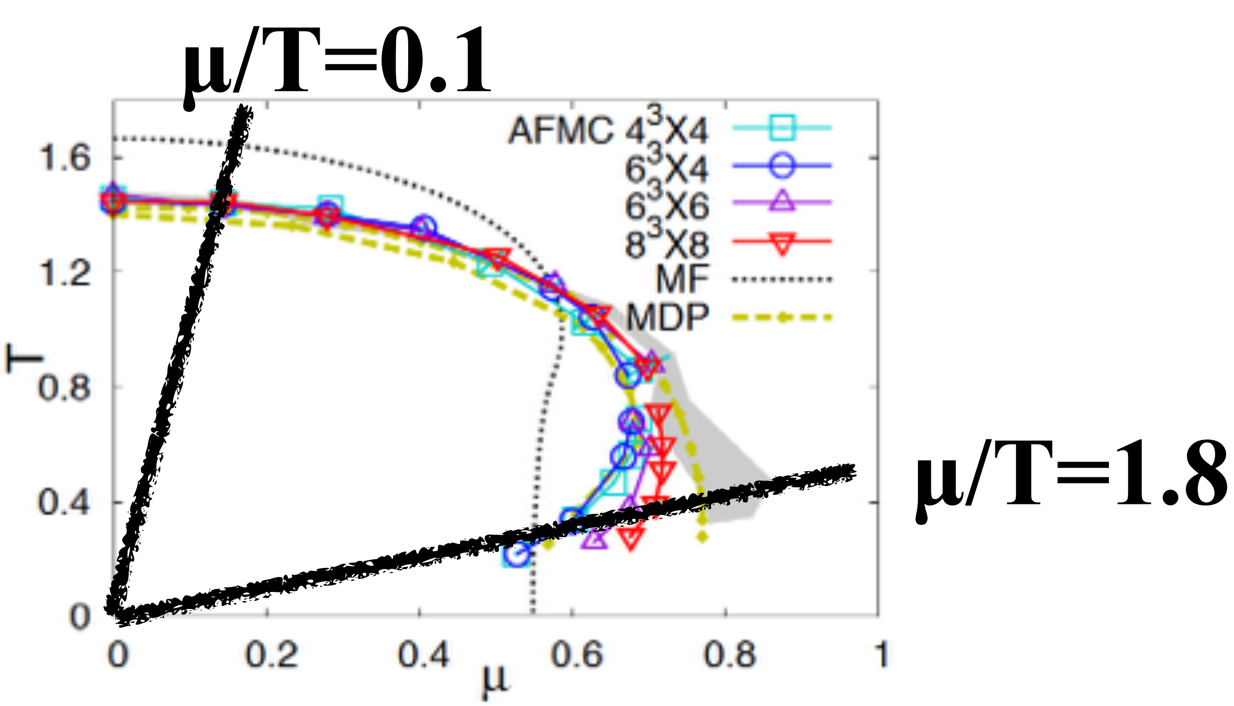
Phase diagram is consistent with MDP

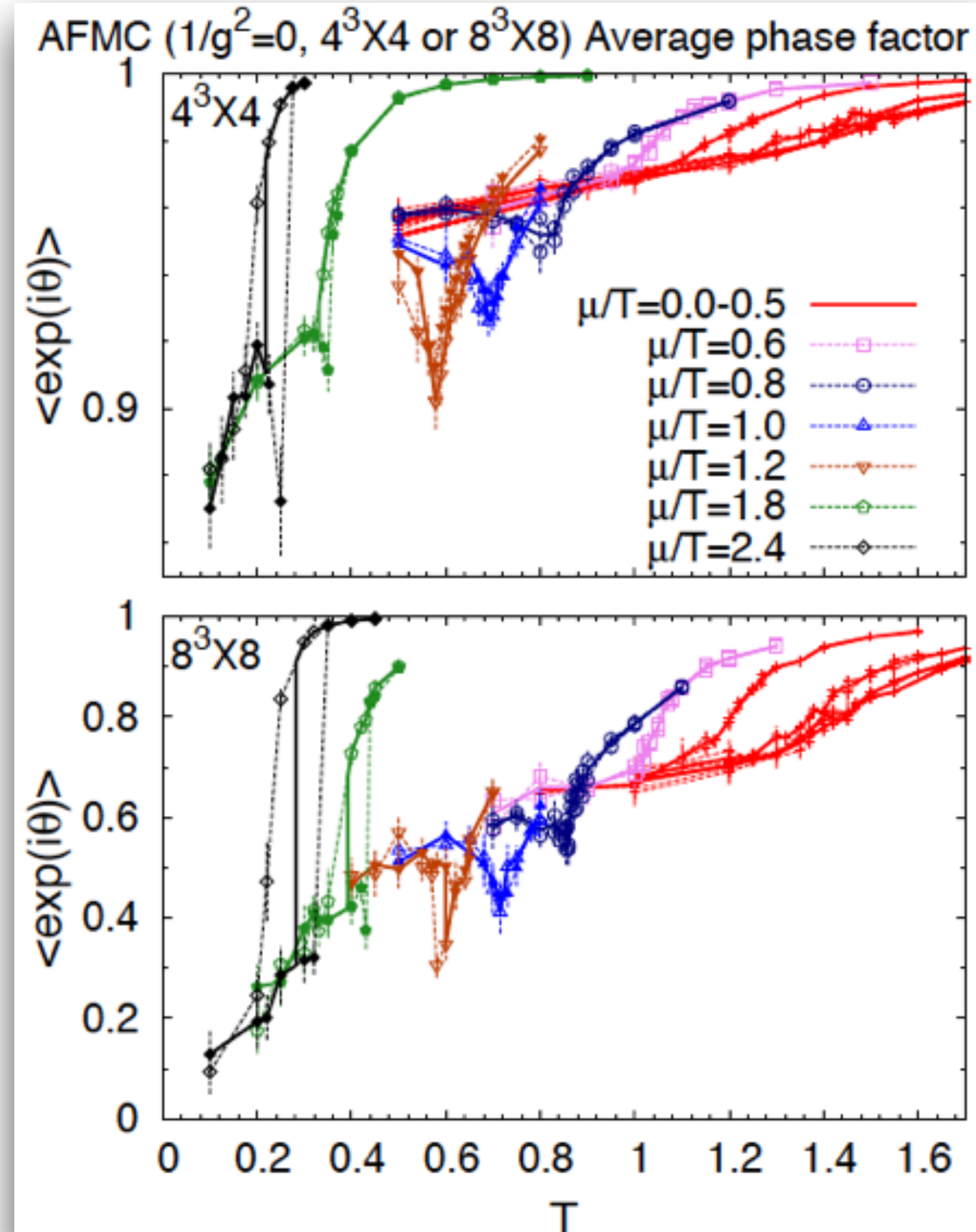
Results - average phase factor in SCL

- Average phase factor
  = Weight cancellation

$$\langle e^{i\theta} \rangle = Z_{\text{full}} / Z_{\text{phase quenched}}$$

  - 4^4 lattice $\langle e^{i\theta} \rangle \geq 0.85$
  - 8^4 lattice $\langle e^{i\theta} \rangle \geq 0.1$



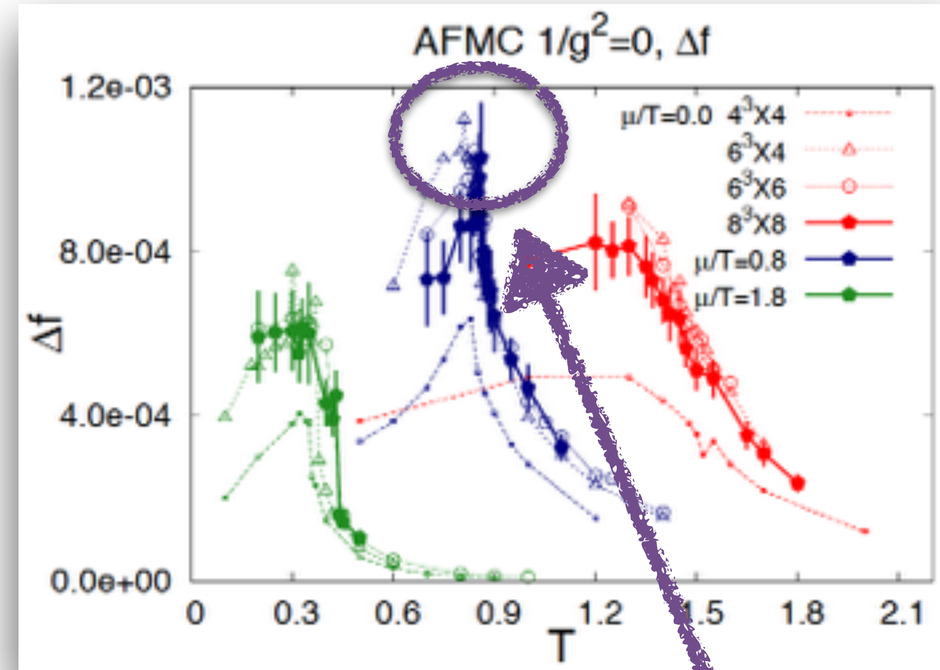


Discussion - the sign problem in SCL

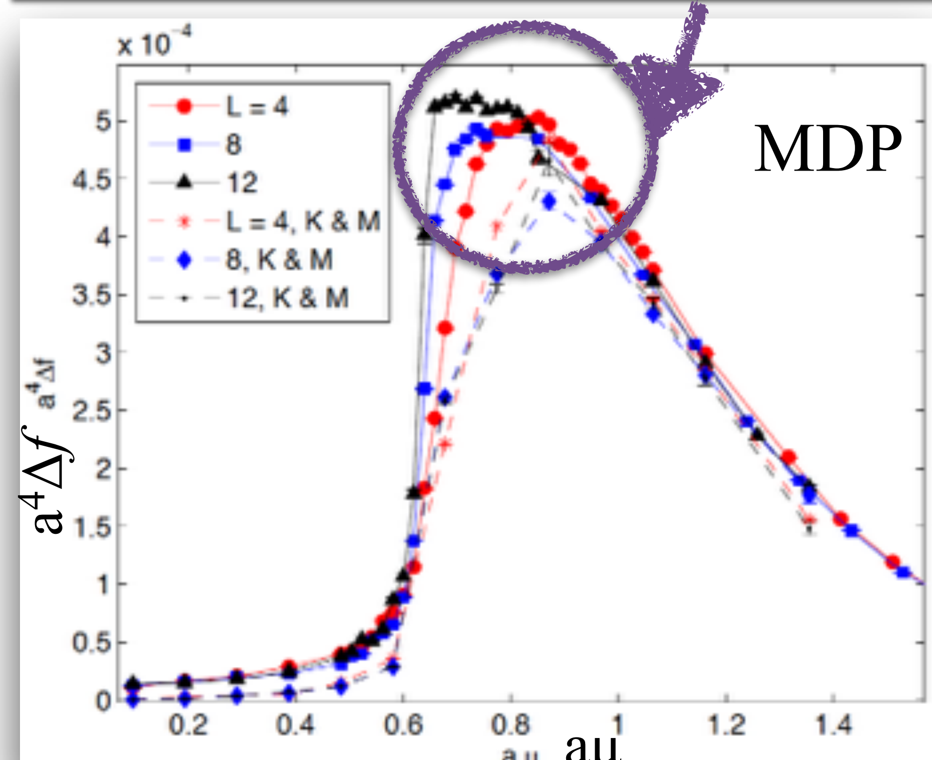
- The severity of the sign problem
  - $\Delta f\,(= f^{\text{full}} - f^{\text{p.q.}})$, the difference of the free energy density in full and phase quenched MC simulations

$$e^{-L^3 N_\tau \Delta f} = Z_{\text{full}}/Z_{\text{p.q.}} = \langle e^{i\theta} \rangle_{\text{p.q.}}$$

    - $\Delta f\,(\text{AFMC}) \simeq 1.0\times10^{-3}$
    - $\Delta f\,(\text{MDP}) \simeq 0.5\times10^{-3}$
  - AFMC has more sever weight cancellation
    - $\Delta f\,(\text{AFMC}) \simeq 2\times\Delta f\,(\text{MDP})$
- Do we need to improve AFMC method for a larger lattice and finite coupling?



Almost the same point in the QCD phase diagram



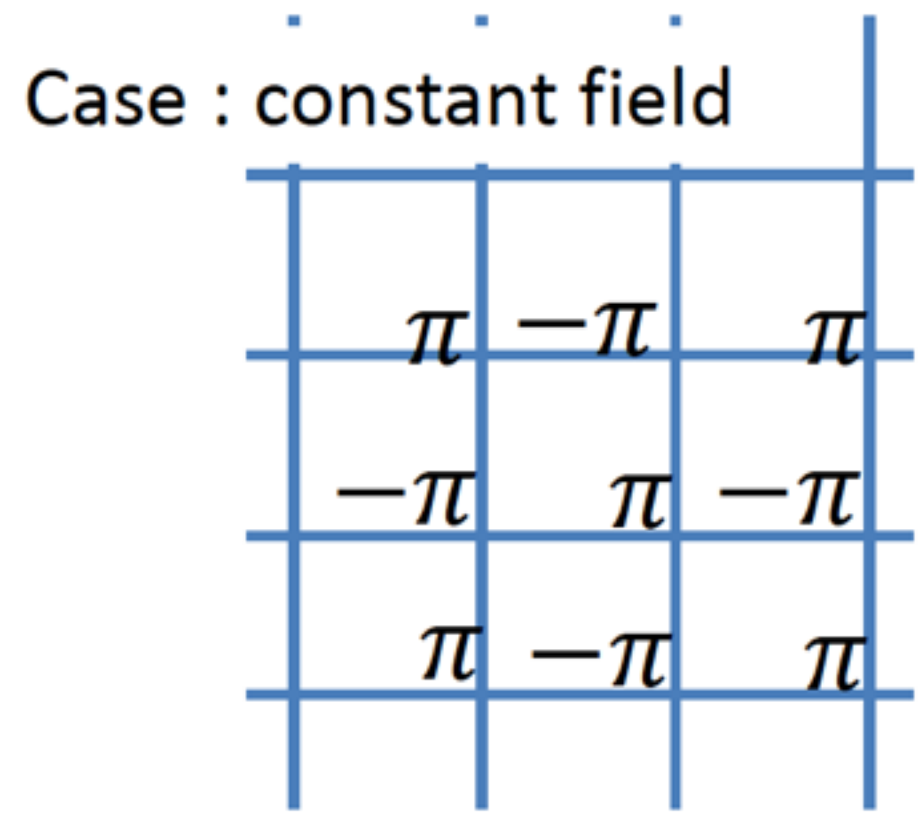
Discussion - source of the sign problem

- Modified mass term

$$m_x = m + \frac{1}{4N_c}\sum_j (\sigma + i\epsilon\pi)_{x\pm\hat{j}}$$

$$\epsilon_x = (-1)^{x_0+\cdots+x_d}$$

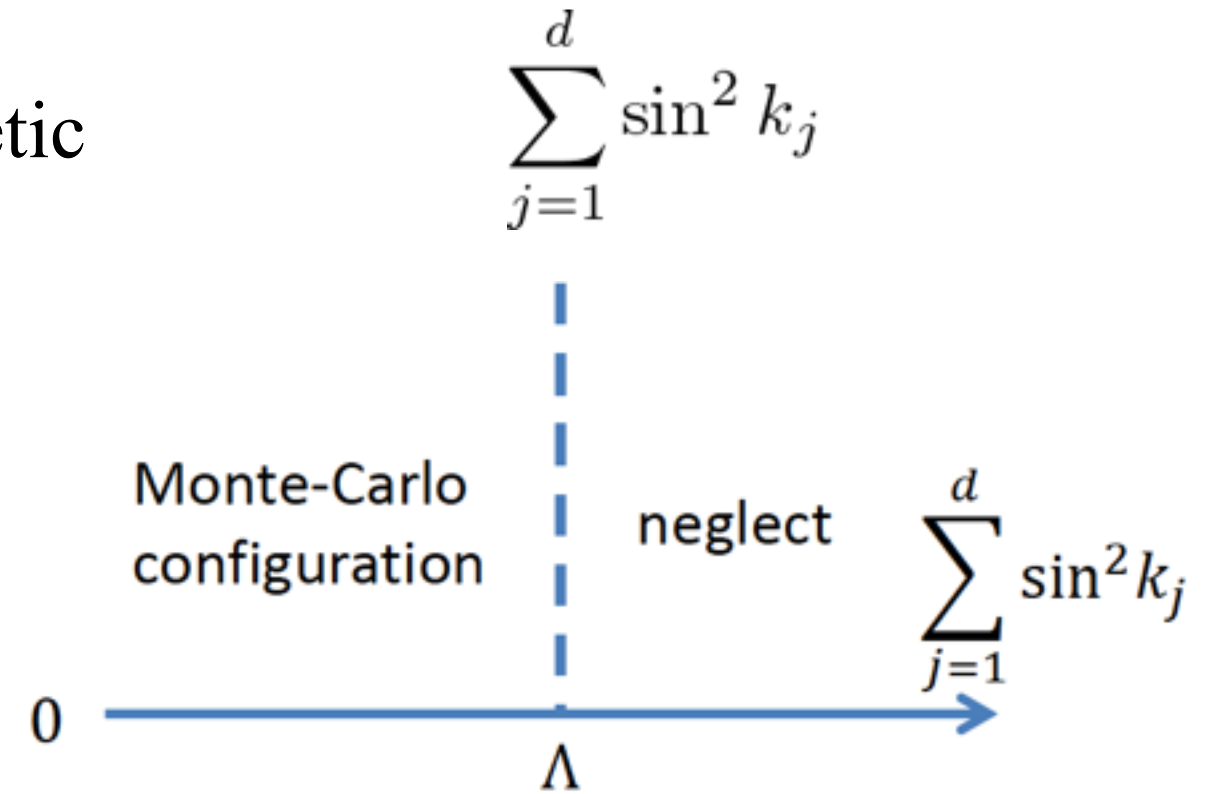
- Momentum
  - Low momentum
    - Cancellation mechanism
    - small phase
  - High momentum
    - No cancellation mechanism

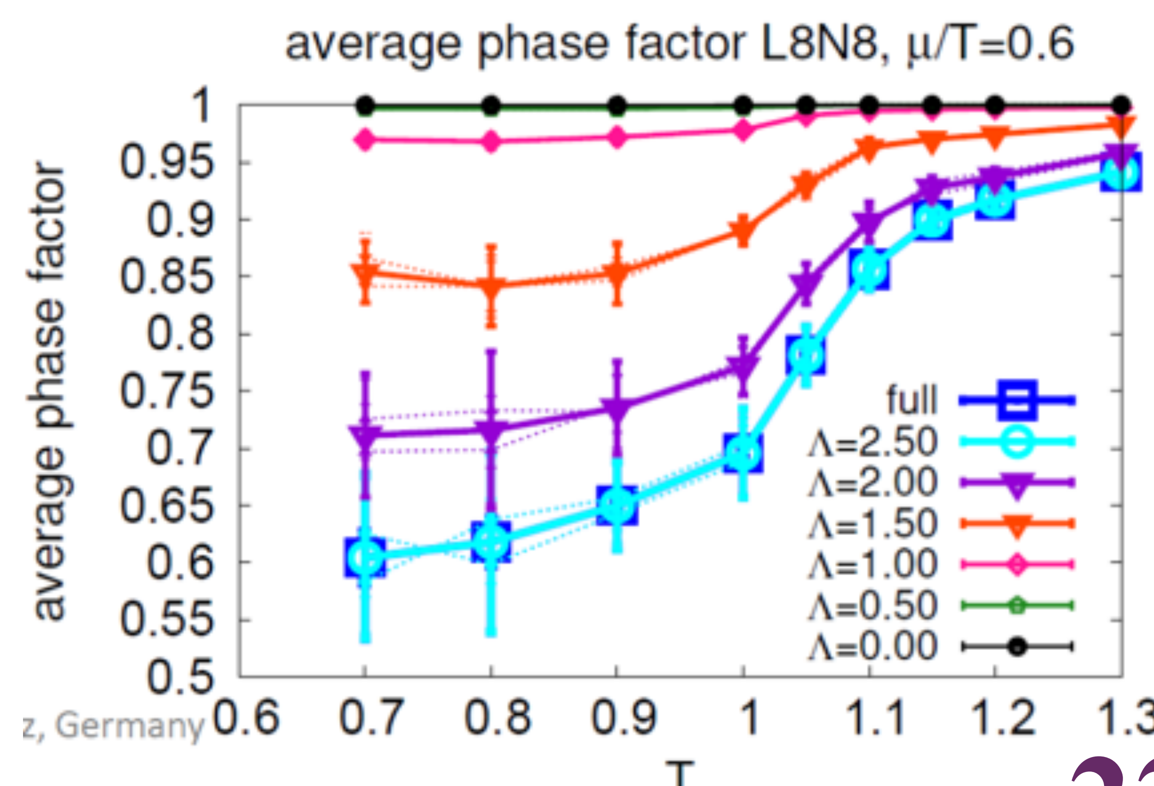


Discussion - auxiliary field momentum cut-off

- High momentum
  = High momentum modes of spatial kinetic momentum
- Cutting off high momentum auxiliary field components
  - Reductions of weight cancellations?
- Qualitative confirmations
  - Average phase factor goes to 1
  - Weight cancellations weaken

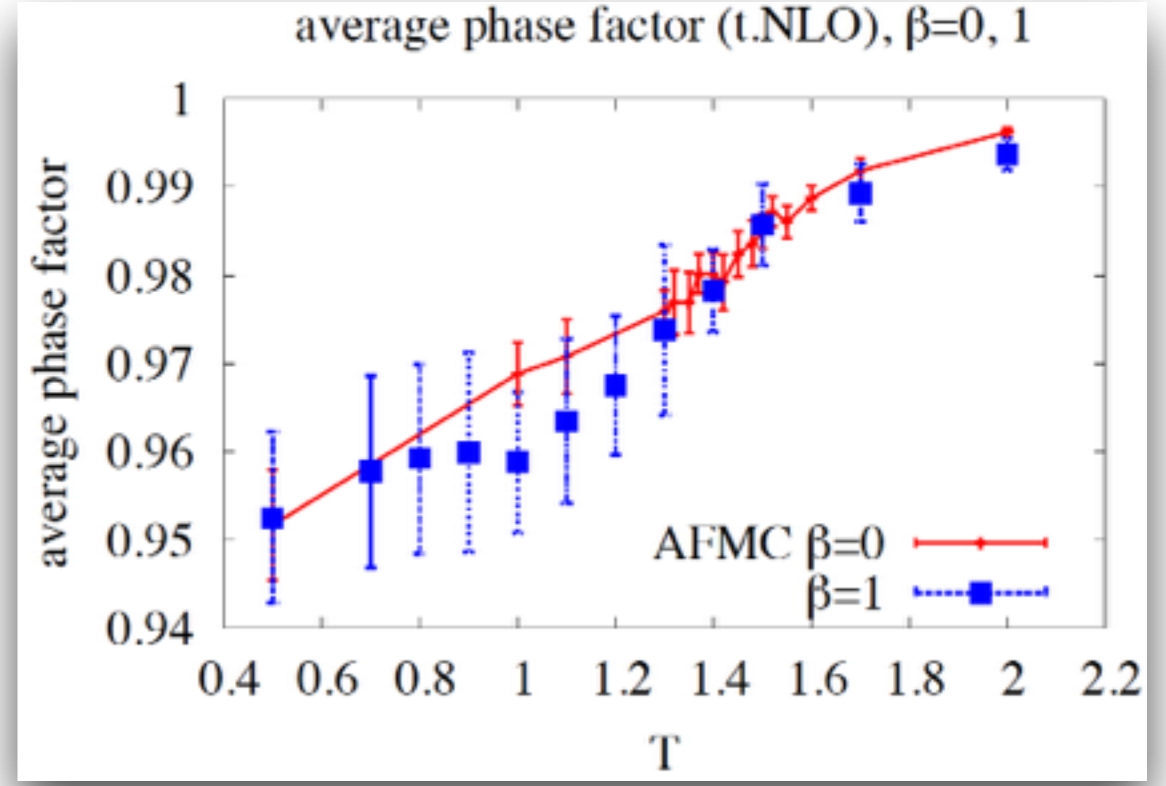
e.g. $8^3{\times}8$ lattice, μ/T=0.6

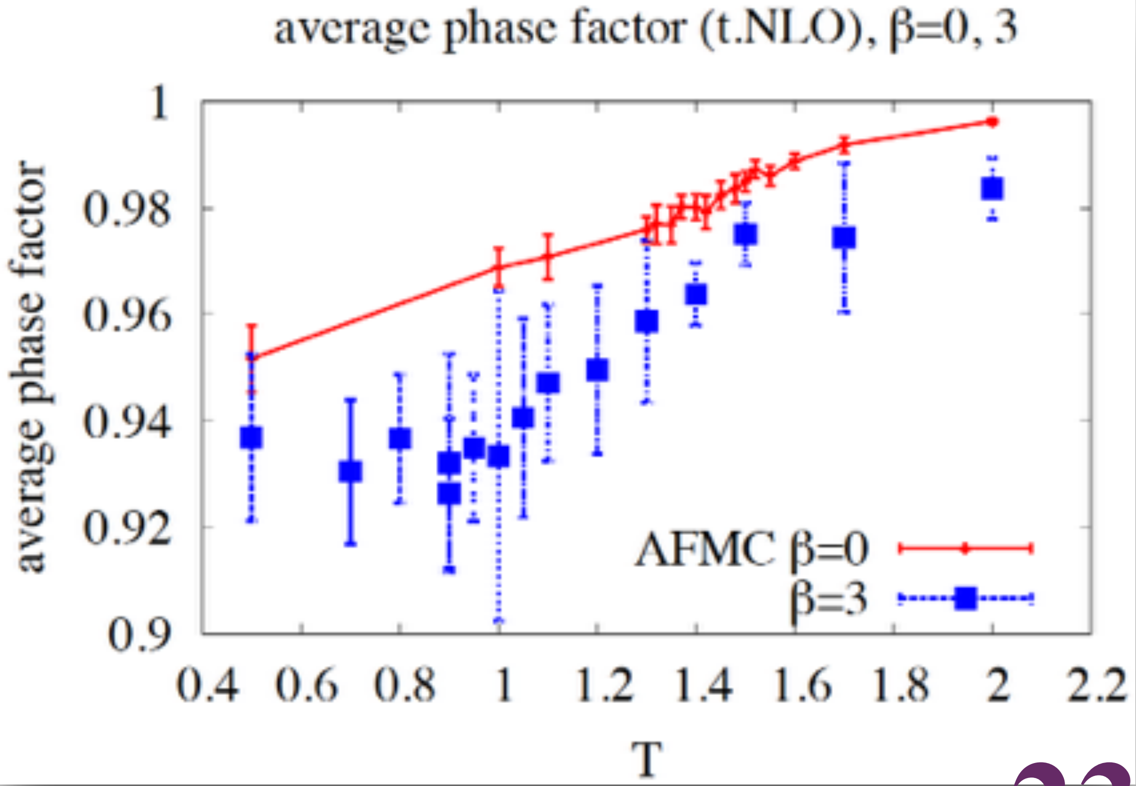




Results - temporal NLO (t.NLO) effects - (1)

- Average phase factor (β=0,1,3)
  - Large enough $\langle e^{i\theta} \rangle \geq 0.9$
    - sign problem is not serious in small lattice
    - t.NLO auxiliary fields do not drastically affect average phase factor at μ=0

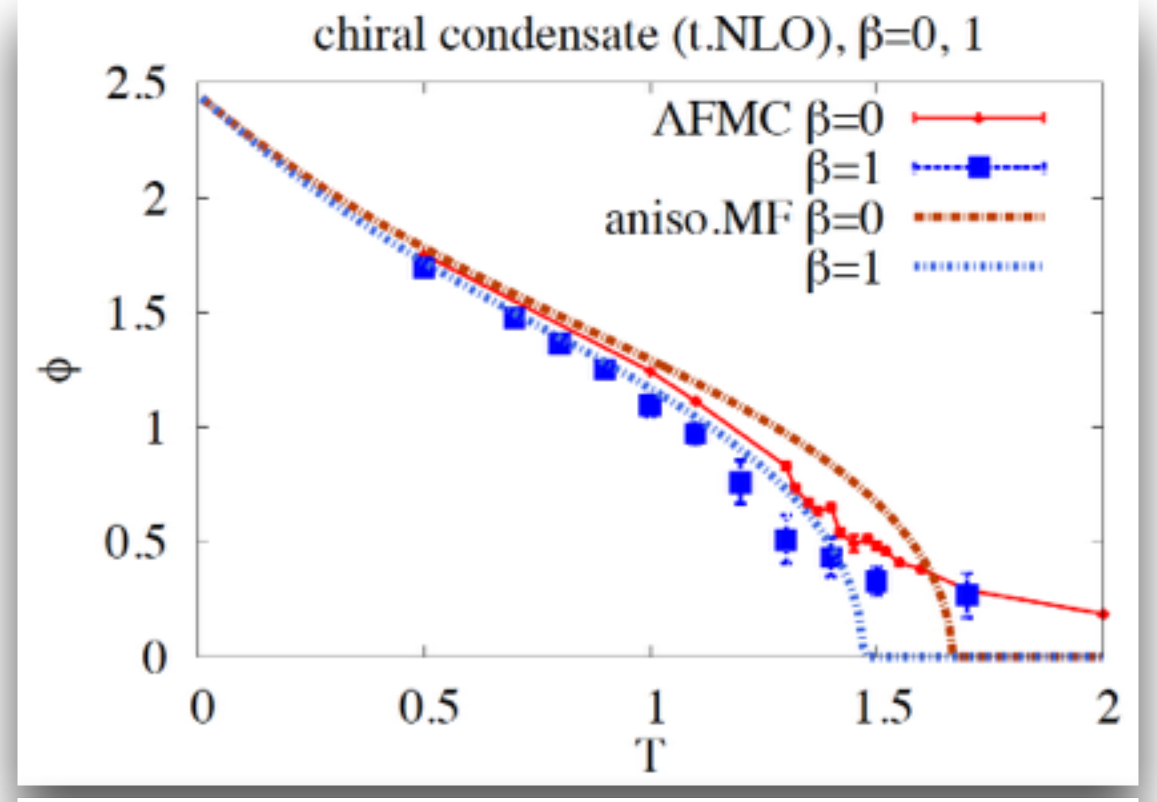


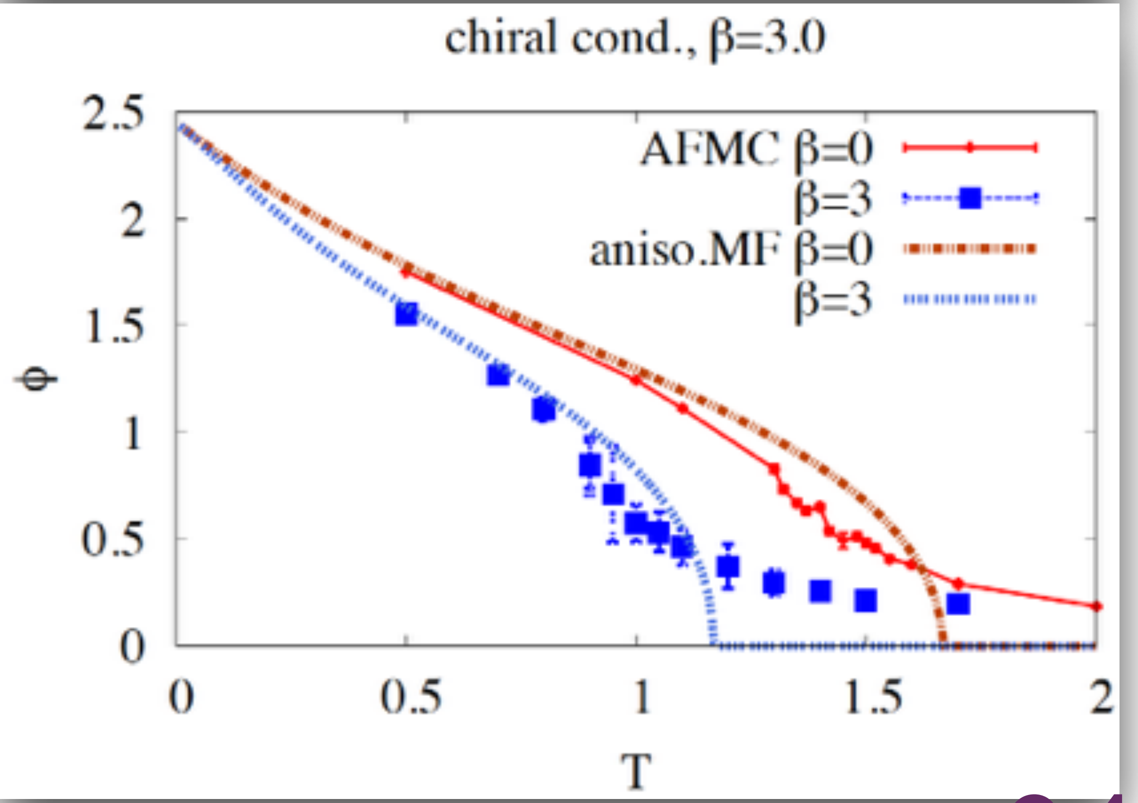


Results - temporal NLO (t.NLO) effects - (2)

- Chiral condensate (Chiral radius) (β=0,1,3)
  - Fluctuation reduces chiral condensate compared with mean field (MF) results.
  - t.NLO auxiliary fields reduce chiral condensate compared with SCL results.
    - t.NLO AFs generate wave functional renormalization, which rescale effective mass.

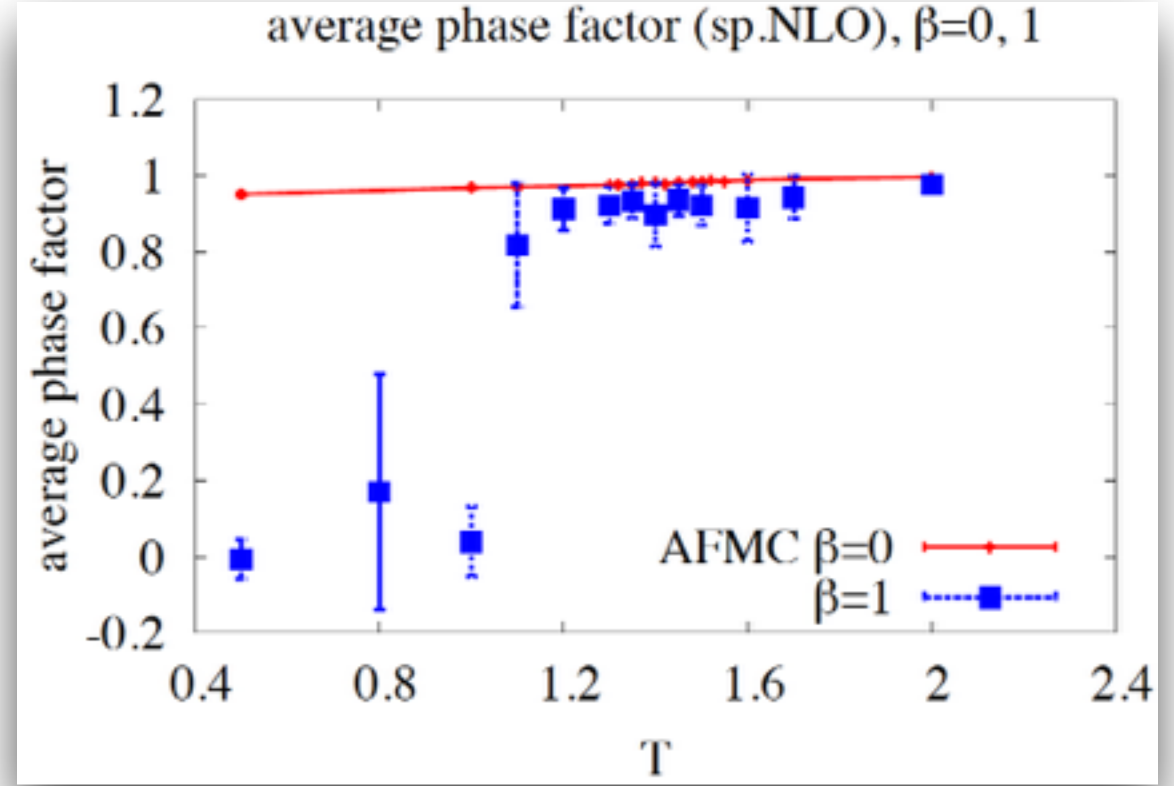
Miura, Nakano, Ohnishi, Kawamoto (2009)
Nakano, Miura, Ohnishi (2011)

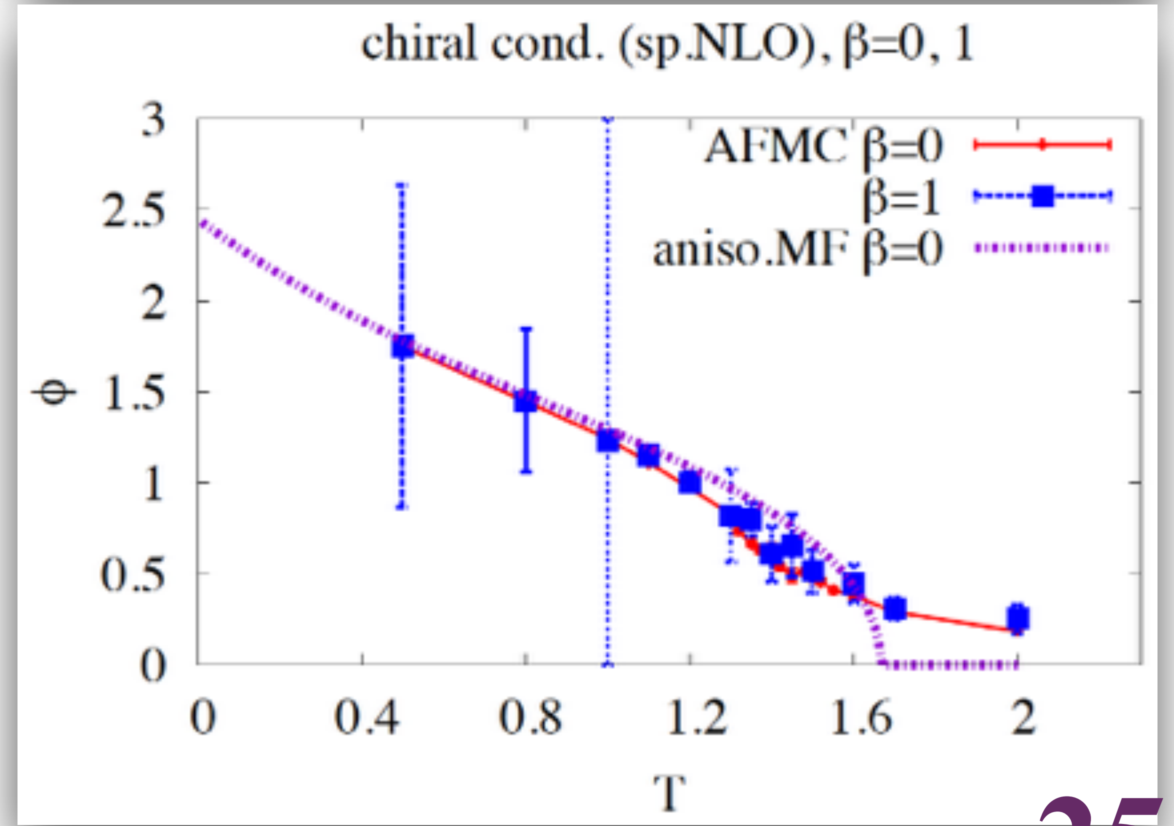




Results - spatial NLO (sp.NLO) effects

- Average phase factor (β=0.1)
  - Smaller than average phase factor of temporal NLO and SCL results
- Chiral condensate
  - almost the same as β=0 up to current β
  - similar to aniso. MF analysis



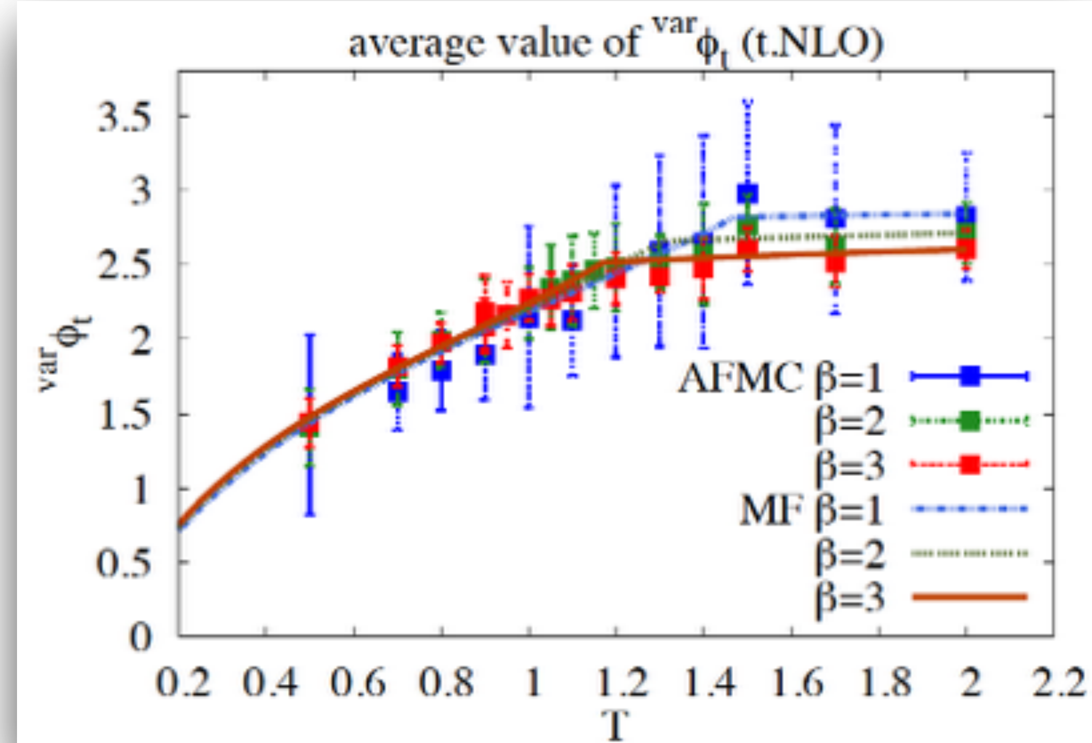


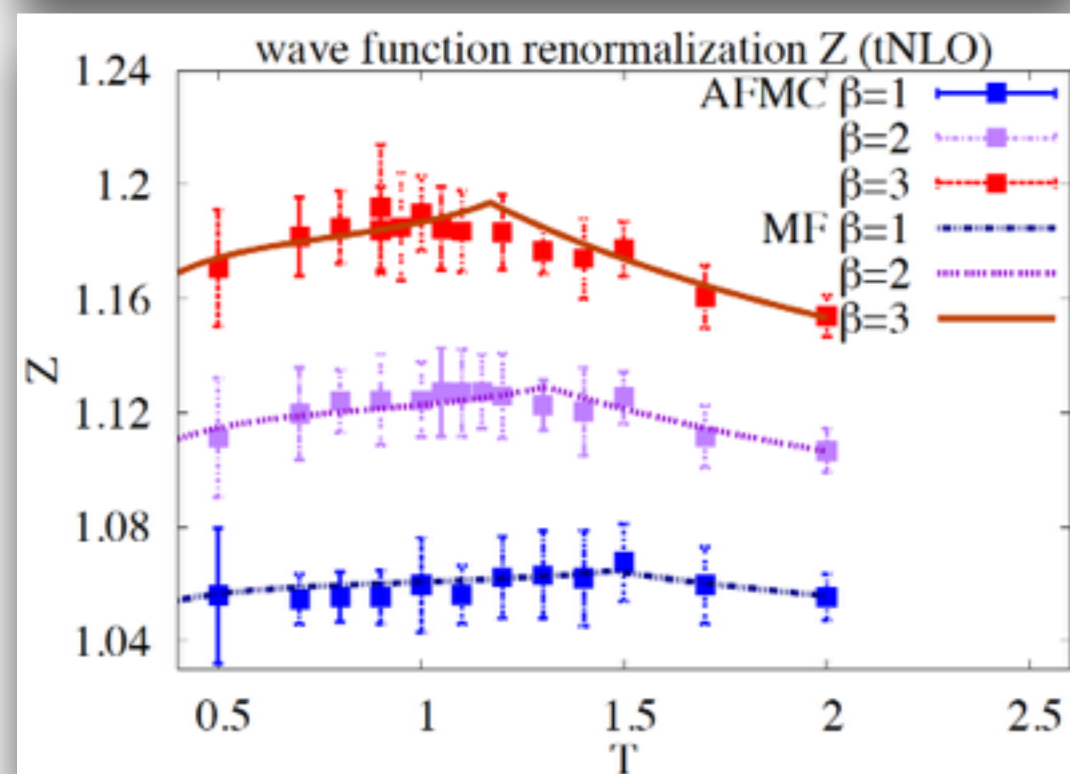
Summary

- We give an effective action including both finite coupling and fluctuation effects.
- In SCl, we give results of order parameters, phase diagram, and discus the origin of the sign problem
  - 1st order phase transition at high μ, 2nd or crossover at low μ
  - Sign problem comes from high momentum modes of the pion field
- We give results of NLO effects
  - From numerical results at μ=0,
    - chiral condensate
      - is reduced by temporal NLO fields
      - is not altered much by spatial NLO fields
    - average phase factor
      - is large enough with temporal NLO fields
      - becomes small with spatial NLO fields
- We are developing a new way to weaken the sign problem to investigate larger μ, β and lattice in AFMC.

Results - t. NLO effects (t.NLO AFs & Z)

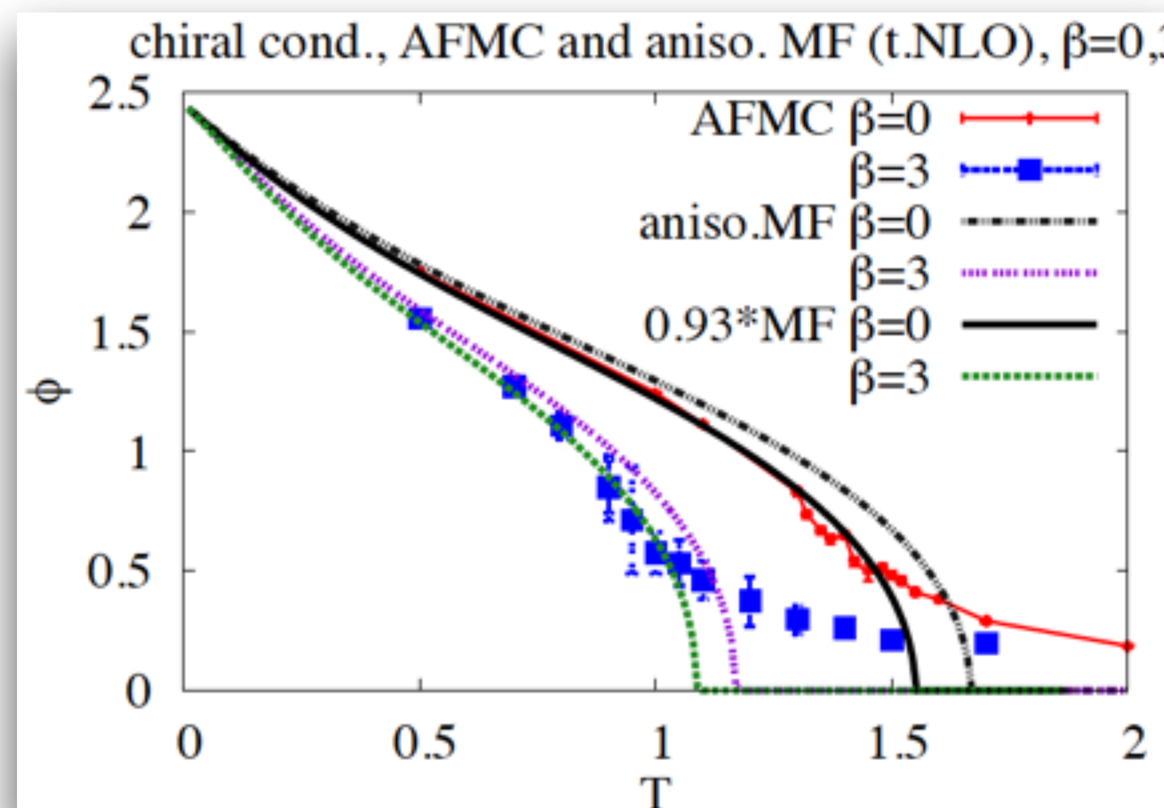
- AFs for t. NLO fields in MF
  - φ_t : $\varphi_t = -\langle V^+ - V^- \rangle /2$
  - ω_t : $\omega_t = -\langle V^+ + V^- \rangle /2 = \rho_q$
    $= 0, \ (\mu = 0)$
- Wave function renormalization Z
  - Z at μ=0 in MF
    $$Z = (1 + \beta_t \varphi_t) \qquad \beta_t = d/N_c^2 g^2$$
  - Rescaling modified mass
    $$S_{\mathrm{eff}}^{\mathrm{EHS}} = \frac{1}{2}\sum_x \Phi_x^2 + \sum_x m_x(\Phi) M_x + \frac{1}{2}\sum_x Z_x(\Phi)\left[V_x^+(\tilde{\mu}(\Phi)) - V_x^-(\tilde{\mu}(\Phi))\right]$$





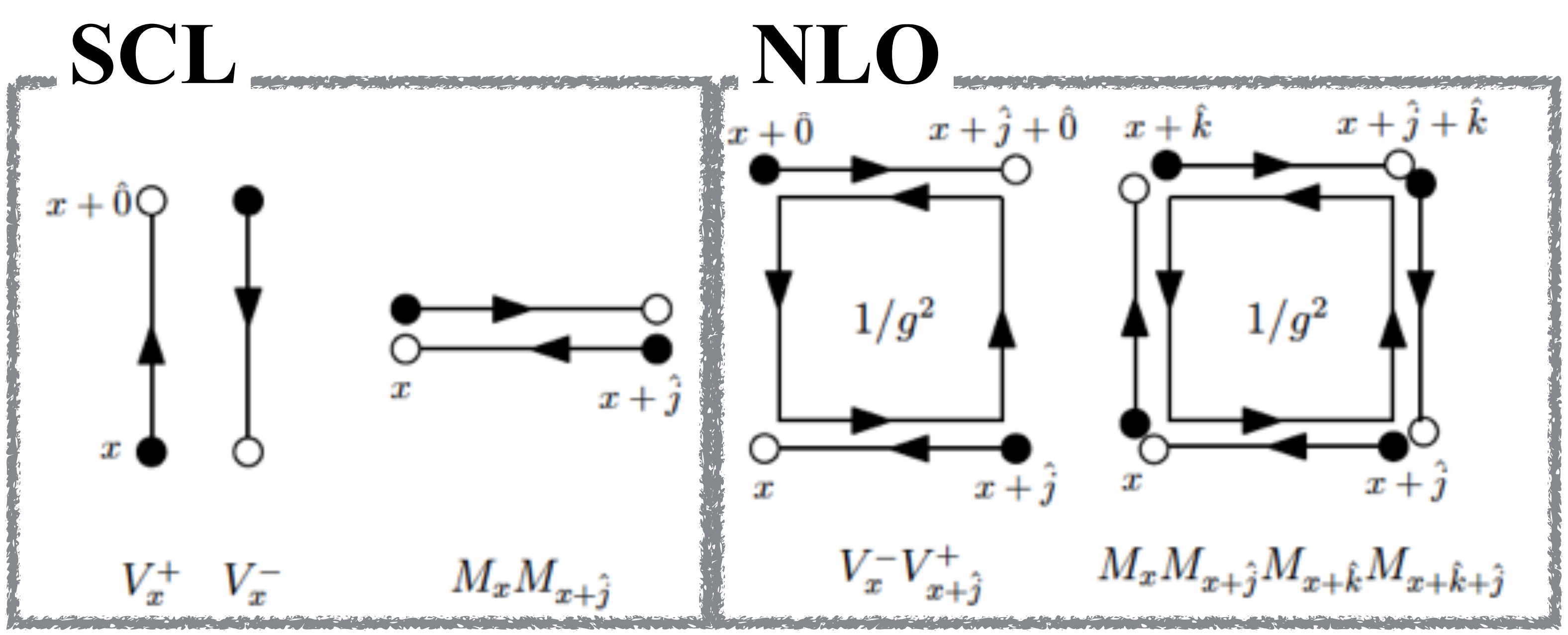
Results - t. NLO effects (chiral condensate)

- Compared with MF results, chiral condensate
  - is reduced by approximately 7% in SCL
  - is also reduced by approximately 7% in t.NLO
- Surprisingly, chiral condensate is altered cumulatively by finite coupling and fluctuation effects.



NLO effective action (1)

- Auxiliary Fields (AFs)
  - SCL : σ and π are AFs for M terms
  - spatial NLO : Σ and Π are AFs for MM terms
  - temporal NLO : ω and Ω are AFs for V terms



σ, π ω, Ω Σ, Π

NLO effective action (2)

- Correction to mass, μ, wave function

$$m_x = m_0 + \boxed{\frac{1}{4N_c}\sum_j \left[(\sigma + i\epsilon\pi)_{x-\hat{j}} + (\sigma + i\epsilon\pi)_{x+\hat{j}}\right]}$$ **SCL**

$$\boxed{+ C_s i \left[(\varphi_x - i\phi_x) + \sum_j \left(C^s_{j,x-\hat{j}}\varphi_{x-\hat{j}} + iC^s_{j,x-\hat{j}}\phi_{x-\hat{j}}\right)\right]}$$ **sp. NLO**

$$e^{\tilde{\mu}_x} = e^{\mu} e^{-\delta\mu_x} = e^{\mu}\boxed{\sqrt{\alpha_x^- / \alpha_x^+}}$$ **t. NLO**

$$Z_x = \boxed{\sqrt{\alpha_x^+ \alpha_x^-}}$$ **t. NLO**

$$C_\tau = 1/(2N_c^2 g^2 \gamma)$$

$$C_s = 1/(2N_c^3 g^2 \gamma)$$

$$C^s_{j,x} = C^s_{j,x}(\Sigma, \Pi)$$

$$\alpha_x^- = 1 + C_\tau \sum_j \left[i\omega_{x\pm\hat{j}} + (\epsilon\Omega)_{x\pm\hat{j}}\right]$$

$$\alpha_x^+ = 1 - C_\tau \sum_j \left[i\omega^*_{x\pm\hat{j}} + (\epsilon\Omega^*)_{x\pm\hat{j}}\right]$$

NLO effective action (3)

- Effective action

$$S_{\text{eff}}^{(\text{NLO})} = \frac{L^3 C_s}{8N_c} \sum_{\tau,\boldsymbol{u},\kappa_u^j>0,j} \kappa_u^{(j)} \left[|\Sigma_{\boldsymbol{u}}^{(j)}|^2 + |\Pi_{\boldsymbol{u}}^{(j)}|^2 \right] + L^3 C_\tau \sum_{\tau,\boldsymbol{k},f(\boldsymbol{k})>0} f(\boldsymbol{k}) \left[|\omega_{\boldsymbol{k},\tau}|^2 + |\Omega_{\boldsymbol{k},\tau}|^2 \right]$$
$$+ \frac{L^3}{4N_c} \sum_{\boldsymbol{k},\tau,f(\boldsymbol{k})>0} f(\boldsymbol{k}) \left[|\sigma_{\boldsymbol{k},\tau}|^2 + |\pi_{\boldsymbol{k},\tau}|^2 \right] + \frac{C_s}{4N_c} \sum_x \left[\phi_x^2 + \varphi_x^2 \right]$$
$$- \sum_{\boldsymbol{x}} \log \left[X_{N_\tau}(\boldsymbol{x})^3 - 2\hat{Z}(\boldsymbol{x})^2 X_{N_\tau} + \hat{Z}(\boldsymbol{x})^3 2\cosh\left(3\hat{\tilde{\mu}}(\boldsymbol{x}) \right) \right] .$$

$$C_\tau = 1/(2N_c^2 g^2 \gamma)$$
$$C_s = 1/(2N_c^3 g^2 \gamma)$$
$$C_{j,x}^s = C_{j,x}^s(\Sigma, \Pi)$$
$$\alpha_x^- = 1 + C_\tau \sum_j \left[i\omega_{x\pm\hat{j}} + (\epsilon\Omega)_{x\pm\hat{j}} \right]$$
$$\alpha_x^+ = 1 - C_\tau \sum_j \left[i\omega_{x\pm\hat{j}}^* + (\epsilon\Omega^*)_{x\pm\hat{j}} \right]$$

$$f(\boldsymbol{k}) = \sum_{j>0} \cos k_j$$
$$\kappa_u^{(j)} = \sum_{k(\neq j)} \cos u_k$$
$$e^{\tilde{\mu}_x} = e^{\mu} \sqrt{\alpha_x^- / \alpha_x^+}$$
$$\hat{Z}(\boldsymbol{x}) = \prod_i Z_{\boldsymbol{x},i}$$

X_N is a known function

NLO effective action (4)

- Calculation of fermion determinant

Faldt, Petersson (1986)

$$\mathcal{R} \equiv \int \mathcal{D}\left[\chi, \bar{\chi}, U_0\right] e^{-\sum_{\boldsymbol{x},\boldsymbol{y}} \bar{\chi}_{\boldsymbol{x}} G^{-1}_{\boldsymbol{x},\boldsymbol{y}} \chi_{\boldsymbol{y}}}$$

$$= \prod_{\boldsymbol{x}} \int \mathcal{D}U_{0,\boldsymbol{x}} \begin{vmatrix} I_1 \cdot \mathbf{1}_{N_c} & \alpha_1 \cdot \mathbf{1}_{N_c} & 0 & \cdots & \beta_{N_\tau} U^{+}_{0,\boldsymbol{x}} \\ -\beta_1 \cdots \mathbf{1}_{N_c} & I_2 \cdot \mathbf{1}_{N_c} & \alpha_2 \cdot \mathbf{1}_{N_c} & \cdots & 0 \\ 0 & & \ddots & \ddots & \vdots \\ 0 & & & & \alpha_{N_\tau - 1} \cdot \mathbf{1}_{N_c} \\ -\alpha_{N_\tau} U_{0,\boldsymbol{x}} & & 0 & -\beta_{N_\tau - 1} \cdot \mathbf{1}_{N_c} & I_{N_\tau} \cdot \mathbf{1}_{N_c} \end{vmatrix}$$

$$= \prod_{\boldsymbol{x}} \int \mathcal{D}U_{0,\boldsymbol{x}} \mathrm{det}_{N_c} \left[X_{N_\tau} \cdot \mathbf{1}_{N_c} + \hat{\beta} U^{+}_{0,\boldsymbol{x}} + (-1)^{N_\tau} \hat{\alpha} U_{0,\boldsymbol{x}} \right] ,$$

$$G^{-1}_{x,y} = \frac{\delta_{\boldsymbol{x},\boldsymbol{y}}}{2} \left[Z_x \left(e^{\tilde{\mu}(x)} U_{x,0} \delta_{x+\hat{0},y} - e^{-\tilde{\mu}(y)} U^{+}_{x,0} \delta_{x-\hat{0},y} \right) + I_x \right] \qquad I = 2m_x/\gamma$$

$$\alpha_i = Z_{\boldsymbol{x},i} e^{\tilde{\mu}_i} \ , \ \beta_i = Z_{\boldsymbol{x},i} e^{-\tilde{\mu}_i} \ , \ \gamma_i = \alpha_i \beta_i = Z^2_{\boldsymbol{x},i}$$

$$\hat{\alpha} = \alpha_1 \alpha_2 \cdots \alpha_{N_\tau} = \hat{Z} e^{\hat{\tilde{\mu}}(\boldsymbol{x})} \qquad \hat{\beta} = \beta_1 \beta_2 \cdots \beta_{N_\tau} = \hat{Z} e^{-\hat{\tilde{\mu}}(\boldsymbol{x})} \quad \hat{\alpha}\hat{\beta} = \hat{Z}(\boldsymbol{x})^2$$

NLO effective action (5)

- Calculation of fermion determinant

Faldt, Petersson (1986)

$$\mathcal{R} = \prod_{\boldsymbol{x}} \left[X_{N_\tau}(\boldsymbol{x})^3 - 2\hat{Z}(\boldsymbol{x})^2 X_{N_\tau} + \hat{Z}(\boldsymbol{x})^3 2\cosh\left(3\hat{\tilde{\mu}}(\boldsymbol{x})\right)\right]$$

- X_N : $X_{N_\tau}(I_1,\cdots,I_{N_\tau};\gamma_1,\cdots,\gamma_{N_\tau}) = B_{N_\tau}(I_1,\cdots,I_{N_\tau};\gamma_1,\cdots,\gamma_{N_\tau-1}) + \gamma_{N_\tau} B_{N_\tau-2}(I_2,\cdots,I_{N_\tau-1};\gamma_2,\cdots,\gamma_{N_\tau-2})$,

$$B_{N_\tau}(I_1,\cdots,I_{N_\tau};\gamma_1,\cdots,\gamma_{N_\tau-1}) = I_{N_\tau} B_{N_\tau-1}(I_1,\cdots,I_{N_\tau-1};\gamma_1,\cdots,\gamma_{N_\tau-2}) + \gamma_{N_\tau-1} B_{N_\tau-2}(I_1,\cdots,I_{N_\tau-2};\gamma_1,\cdots,\gamma_{N_\tau-3})$$

$$B_N(I_1,\ldots,I_N;\gamma_1,\ldots,\gamma_{N-1}) = \begin{vmatrix} I_1 & \alpha_1 & 0 & 0 & \cdots & & 0 \\ -\beta_1 & I_2 & \alpha_2 & 0 & & & 0 \\ 0 & -\beta_2 & I_3 & \alpha_3 & & & 0 \\ 0 & 0 & -\beta_3 & I_4 & & & \\ & & & & \ddots & & \alpha_{N-1} \\ 0 & 0 & 0 & 0 & \cdots & -\beta_{N-1} & I_N \end{vmatrix}$$